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International Journal of Heat and Fluid Flow 27 (2006) 671–683

International Journal of HEAT AND FLUID FLOW

www.elsevier.com/locate/ijhff

Nonlinear eddy diffusivity models reflecting buoyancy effect for wall-shear flows and heat transfer

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Available online 31 March 2006

Abstract

The main objective of this study is to construct new nonlinear eddy diffusivity models reflecting buoyant effects in wall-bounded turbulent shear flows and heat transfer. It is now well known that the turbulent heat-fluxes, which are the key quantities for the prediction of turbulent flows with buoyancy, are not modeled accurately by employing the conventional turbulence models using eddy diffusivities. In order to appropriately predict wall-bounded turbulent flows with buoyancy, an innovative turbulence heat-transfer model with eddy diffusivities which are composed of the k–ε two-equation model for velocity field and the k_θ–ε_θ two-equation model for thermal field, must be constructed. Consequently, we should improve the modeled expressions for Reynolds stresses and turbulent heat-fluxes reflecting the buoyant effect in wall-bounded turbulent shear flows. The existing two-equation turbulence models were evaluated on the basis of the DNS data of channel flows with buoyancy. Using the results of evaluation, we constructed new modeled expressions for Reynolds stresses and turbulent heat-fluxes in explicit algebraic models, and reconstructed the nonlinear two-equation turbulence models for the buoyancy-affected wall-shear flows and heat transfer, including the newly proposed nonlinear eddy diffusivity for a momentum model (NLEDM) and the nonlinear eddy diffusivity for heat model (NLEDHM). The proposed nonlinear two-equation turbulence models reflecting buoyancy effect appropriately predict wall-bounded turbulent shear flows with buoyancy given by DNS.

Keywords: Nonlinear eddy diffusivity model; Two-equation turbulence model; Buoyant effect; Turbulence; Heat transfer; Explicit algebraic model

1. Introduction

The main objective of this study is to construct nonlinear two-equation turbulence models reflecting the buoyant effect in wall-bounded turbulent shear flows and heat transfer. Wall-bounded turbulent shear flows with buoyancy have been encountered in many engineering-relevant applications such as a flow in a computer or room. Two-equation turbulence heat-transfer models for analysis of forced convection have been developed to accurately calculate forced convective turbulence flows (e.g., Nagano and Hattori, 2003; Hattori and Nagano, 1998; Abe et al., 1996; Nagano and Shimada, 1996). Since the buoyancy effect is hard to reflect properly in the modeled expressions of Reynolds shear stress and turbulent heat-flux using the linear eddy diffusivities, two-equation turbulence heat-transfer models might not be adequate for an analysis of turbulent shear flow with buoyancy. For example, although the transport equations for Reynolds shear stress and turbulent heat-flux have a buoyant term, respectively, reflecting adequately this term in relevant eddy diffusivities might be difficult. On the other hand, in order to predict adequately wall-bounded turbulent flow with buoyancy, a conventional two-equation heat-transfer models might not be appropriate, because the streamwise turbulent heat-flux needed to predict the flow cannot be calculated by the model for the model definition. Murakami et al. (1996) proposed linear eddy diffusivity models (EDMs) for buoyancy-affected flow. Here, the buoyant effect is included in the buoyant-reflected model function. However, this function has to be conditionally applied in the

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0142-727X/$ - see front matter
doi:10.1016/j.ijheatfluidflow.2006.02.012

Nomenclature

Ar	Archimedes number, $g\beta\Delta\Theta L_0/U_0$
b_{ij}	anisotropy tensor, $\overline{u_iu_j}/2k-\delta_{ij}/3$
c_p	specific heat at constant pressure
C_0	model constant
C_D, C_η	constants in nonlinear model
C_{D1}, C_{D2}, C_{P1}, C_{P2}, C_{P3}	constants in transport equation for ε_θ
C_λ, C_m	constants in eddy diffusivity for heat
C_μ	constant in eddy viscosity
$C_{\varepsilon1}$, $C_{\varepsilon2}$, $C_{\varepsilon3}$	constants in transport equation for ε
f_B	function to guarantee realizability of anisotropy tensor
$f_{D1}, f_{D2}, f_{P1}, f_{P2}, f_{P3}$	functions in transport equation for ε_θ
f_R	function of time-scale ratio
$f_w(\xi)$	wall-reflection function
f_ε	low-Reynolds-number model function in transport equation for ε
f_λ	low-Reynolds-number model function in eddy diffusivity for heat
f_μ	low-Reynolds-number model function in eddy viscosity
k	turbulence energy, $\overline{u_iu_i}/2$
k_θ	temperature variance, $\overline{\theta^2}/2$
g_i	gravitational acceleration
Gr	Grashhof number, $g\beta\Delta\Theta(2\delta)^3/\nu^2$
n	local coordinate normal to wall surface
n^*	nondimensional distance between point and nearest point on whole surface, $u_\varepsilon n/\nu$
$\overline{P}$	mean static pressure
Pr	Prandtl number, ν/α
R	time-scale ratio, τ_θ/τ_u
Re_b	Reynolds number based on bulk velocity, $2\overline{U}_b\delta/\nu$
Re_τ	Reynolds number based on friction velocity, $u_\tau\delta/\nu$
R_t	turbulent Reynolds number, $k^2/\nu\varepsilon$
S_{ij}	strain-tensor, $(\partial\overline{U}_i/\partial x_j+\partial\overline{U}_j/\partial x_i)/2$
t_τ	friction temperature, $qw/(\rho c_p u_\tau)$
$\overline{U}, \overline{V}, \overline{W}$	mean velocity in x-, y- and z-directions, respectively
u,v,w	turbulent fluctuation in x-, y- and z-directions, respectively
$\overline{U}_b$	bulk velocity
$\overline{U}_i$	mean velocity in x_i-direction
u_i	turbulent fluctuation in x_i-direction
u_τ	friction velocity, $\sqrt{\tau_w/\rho}$
u_ε	Kolmogorov velocity scale, $(\nu\varepsilon)^{1/4}$
x_i	Cartesian coordinate in i-direction
x, y, z	Cartesian coordinate in streamwise, wall-normal and spanwise directions, respectively
y^+	nondimensional distance from wall surface, $u_\tau y/\nu$

Greek symbols

α, α_t	molecular diffusivity and eddy diffusivity for heat
α^t_{ij}	anisotropy eddy diffusivity for heat
β	coefficient of volume expansion
δ	half-width of channel
δ_{ij}	Kronecker delta
$\Delta\Theta$	temperature difference between local and reference points, $\overline{\Theta}_1-\overline{\Theta}_r$
ε	dissipation rate of turbulence energy, $\nu\overline{(\partial u_i/\partial x_j)(\partial u_i/\partial x_j)}$
ε_θ	dissipation rate of temperature variance, $\alpha\overline{(\partial\theta/\partial x_j)(\partial\theta/\partial x_j)}$
ν, ν_t	kinematic viscosity and eddy diffusivity for momentum
ρ	density
Ω_{ij}	vorticity tensor, $(\partial\overline{U}_i/\partial x_j-\partial\overline{U}_j/\partial x_i)/2$
τ	characteristic time-scale of turbulence
τ_m	hybrid/mixed time-scale
τ_θ	time-scale of temperature field, $k_\theta/\varepsilon_\theta$
τ_u	time-scale of velocity field, k/ε
τ_w	wall-shear stress
$\overline{\Theta}$	mean temperature
θ	temperature fluctuation
$\overline{(\ )}$	ensemble- or time-averaged value
$(\ )^+$	normalization by inner variables (u_τ, t_τ, ν)
D/Dt	substantial derivative, $\partial/\partial t+\overline{U}_j\partial/\partial x_j$

specific Reynolds shear stresses and turbulent heat-fluxes, i.e., the model function depends on components of turbulent stresses and turbulent heat-fluxes. Also, the improvement of a prediction for a streamwise turbulent heat-flux was not modeled in particular. It is well known, however, that only linear eddy diffusivity models were successful in predicting the turbulent heat-flux in a turbulent natural convection flow along a vertical plate (Yin et al., 1991; hereinafter referred to as the YNT model). The same investigators adopted the empirical equation indicated below for the streamwise turbulent heat-flux in their turbulence model due to the principal problem of the eddy diffusivity model.

$$-\overline{u\theta}=C_h\frac{-\overline{uv}}{k}\overline{v\theta}$$

where C_h is the model constant.

In order to predict the streamwise turbulent heat-flux without the empirical equation, at least a nonlinear eddy diffusivity for heat model (NLEDHM) or an algebraic heat-flux model (AHFM) is necessary. Moreover, in order to reflect adequately the buoyant effect for the modeled Reynolds shear stress, a nonlinear eddy diffusivity for a momentum model (NLEDM) or an algebraic Reynolds stress model (ASM) is also required. Recently, explicit algebraic models for Reynolds stress and turbulent heat-flux (EASM and EAHFM), as well as NLEDM and NLEDHM, have been improved for wall-bounded shear flows (Hattori and Nagano, 2004; Nagano and Hattori, 2003). Therefore, we have to modify the expressions of Reynolds shear stress of NLEDM and turbulent heat-flux of NLEDHM including the buoyant term based on the nonlinear two-equation heat-transfer model (Abe et al., 1996; Nagano and Hattori, 2003). In this study, in order to determine improvements in these modified expressions for the wall-bounded turbulent shear flows with buoyancy, an a priori test method with the aid of DNS result (e.g., Hattori and Nagano, 2004; Nagano and Hattori, 2003) is carried out for the evaluation of model expressions. As a result, we improve the two-equation turbulence models with NLEDM and NLEDHM for the wall-bounded turbulent shear flows with buoyancy.

2. Governing equations

The Reynolds-averaged equations for nonlinear turbulence models in the buoyancy-affected field can be written as follows (Nagano and Hattori, 2003):

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0 \tag{1}$$

$$\frac{D\overline{U}_i}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i u_j}\right) - g_i \beta \Delta \Theta \tag{2}$$

$$\frac{D\overline{\Theta}}{Dt} = \frac{\partial}{\partial x_j}\left(\alpha \frac{\partial \overline{\Theta}}{\partial x_j} - \overline{u_j \theta}\right) \tag{3}$$

where the Boussinesq approximation is used in Eq. 2, and the Einstein summation convention applies to repeated indices.

In an NLEDM and an NLEDHM, modeled expressions for Reynolds stress and turbulent heat-flux are described as follows:

$$\overline{u_i u_j} = \frac{2}{3}k\delta_{ij} - 2C_0 \nu_t S_{ij} + \text{high-order terms} \tag{4}$$

$$\overline{u_j \theta} = -\alpha^t_{jk} \frac{\partial \overline{\Theta}}{\partial x_k} + \text{high-order terms} \tag{5}$$

where $\nu_t = C_\mu f_\mu (k^2/\varepsilon)$ is the eddy diffusivity for a momentum and $\alpha^t_{jk} = -C_{t0}\overline{u_j u_k}\tau_m$ is the anisotropy eddy diffusivity for heat.

The following transport equations of turbulent quantities making up the expressions of Reynolds shear stress and turbulent heat-flux are given:

$$\frac{Dk}{Dt} = \nu \frac{\partial^2 k}{\partial x_j \partial x_j} + T_k + P_k + G_k - \varepsilon \tag{6}$$

$$\frac{D\varepsilon}{Dt} = \nu \frac{\partial^2 \varepsilon}{\partial x_j \partial x_j} + T_\varepsilon + \frac{\varepsilon}{k}(C_{\varepsilon 1} P_k - C_{\varepsilon 2} f_\varepsilon \varepsilon + C_{\varepsilon 3} G_k) \tag{7}$$

$$\frac{Dk_\theta}{Dt} = \alpha \frac{\partial^2 k_\theta}{\partial x_j \partial x_j} + T_{k_\theta} + P_{k_\theta} - \varepsilon_\theta \tag{8}$$

$$\frac{D\varepsilon_\theta}{Dt} = \alpha \frac{\partial^2 \varepsilon_\theta}{\partial x_j \partial x_j} + T_{\varepsilon_\theta} + \frac{\varepsilon_\theta}{k_\theta}(C_{P1} f_{P1} P_{k_\theta} - C_{D1} f_{D1} \varepsilon_\theta) + \frac{\varepsilon_\theta}{k}(C_{P2} f_{P2} P_k - C_{D2} f_{D2} \varepsilon + C_{P3} f_{P3} G_k) \tag{9}$$

where $P_k[= -\overline{u_i u_j}(\partial \overline{U}_i/\partial x_j)]$, $P_{k_\theta}[= -\overline{u_j \theta}(\partial \overline{\Theta}/\partial x_j)]$ are production terms, and $G_k(= -g_j \beta \overline{u_j \theta})$ is a buoyant term. The turbulent diffusion terms, T_k, T_ε, T_{k_θ} and T_{ε_θ}, are modeled individually using the GGDH modeling (Nagano and Hattori, 2003).

3. Evaluations for NLEDHM and NLEDM in buoyancy-affected wall-shear flows

3.1. Derivation for NLEDHM with buoyancy effect

The transport equation of turbulent heat-flux with the buoyant term is given as follows:

$$\frac{D\overline{u_j \theta}}{Dt} = D_{j\theta} + T_{j\theta} + P_{j\theta} + G_{j\theta} + \Phi_{j\theta} - \varepsilon_{j\theta} \tag{10}$$

where $D_{j\theta}$ is a molecular diffusion term, $T_{j\theta}$ is a turbulent diffusion term, $P_{j\theta}$ is a production term, $G_{j\theta}$ is a buoyant term, $\Phi_{j\theta}$ is a pressure–temperature gradient correlation term, and $\varepsilon_{j\theta}$ is a dissipation term, respectively.

The modeled expression of turbulent heat-flux including the buoyant term is derived using the following relation:

$$\frac{Da^*_j}{Dt} = \frac{1}{\sqrt{k}\sqrt{k_\theta}}(P_{j\theta} + \Phi_{j\theta} - \varepsilon_{j\theta} + G_{j\theta}) - \frac{1}{2}a^*_j\left[\frac{\varepsilon}{k}\left(\frac{P_k}{\varepsilon} + \frac{G_k}{\varepsilon} - 1\right) + \frac{\varepsilon_\theta}{k_\theta}\left(\frac{P_{k_\theta}}{\varepsilon_\theta} - 1\right)\right] \tag{11}$$

where $a^*_j[= \overline{u_j\theta}/(k^{1/2}k_\theta^{1/2})]$ is the nondimensional turbulent heat-flux and the diffusive effect is neglected.

In the local equilibrium state, the following relation holds (Abe et al., 1996):

$$\frac{Da^*_j}{Dt} = 0 \tag{12}$$

Regarding $\Phi_{j\theta}$ and $\varepsilon_{j\theta}$, the general linear expression (Launder, 1976) with the buoyant effect is employed:

$$\Phi_{j\theta} - \varepsilon_{j\theta} = -C_{t1}\frac{\overline{u_j\theta}}{\tau_u} + C_{t2}\overline{u_k\theta}\frac{\partial \overline{U}_j}{\partial x_k} + C_{t3}\overline{u_k\theta}\frac{\partial \overline{U}_k}{\partial x_j} + C_{t4} g_i \beta k_\theta \tag{13}$$

In order to obtain the explicit relation for turbulent heat-flux reflecting buoyancy effect, the procedure proposed in the previous studies (Nagano and Hattori, 2003; Abe et al., 1996) is adopted. Thus, the NLEDHM with buoyancy-affected term can be derived as follows (hereinafter referred to as the NLHNB model):

$$\overline{u_j\theta} = -\frac{C_{\theta 1}}{f_{\mathrm{RT}}}\left[\overline{u_ju_k}\tau_{m1}\frac{\partial\overline{\Theta}}{\partial x_k} - \overline{u_\ell u_k}\tau_{m2}^2\left(C_{\theta 2}S_{j\ell} + C_{\theta 3}\Omega_{j\ell}\right)\frac{\partial\overline{\Theta}}{\partial x_k}\right] - \frac{2C_{\theta 4}\tau_{m2}}{f_{\mathrm{RT}}}\{\delta_{jk} - \tau_{m2}[(C_{\theta 2} - C_{\theta 3})S_{jk} + (C_{\theta 2} + C_{\theta 3})\Omega_{jk}]\}g_k\beta k_\theta \tag{14}$$

where $C_{\theta 1}$, $C_{\theta 2}$, $C_{\theta 3}$ and $C_{\theta 4}$ are model constants. In this study, model constants $C_{\theta 1} = 0.18$, $C_{\theta 2} = 0.15$, $C_{\theta 3} = 0.21$ and $C_{\theta 4} = 0.33$ are adopted. The model function f_{RT} is given as follows:

$$f_{\mathrm{RT}} = 1 + \frac{1}{2}\tau_{m2}^2\left[C_{\theta 2}^2(\Omega^2 - S^2) + (C_{\theta 3}^2 - C_{\theta 2}^2)\Omega^2\right] \tag{15}$$

where $S^2 = S_{mn}S_{mn}$ and $\Omega^2 = \Omega_{mn}\Omega_{mn}$.

In order to reflect wall effect, the mixed time-scales τ_{m1} and τ_{m2} were defined as follows (Nagano and Hattori, 2003):

$$\tau_{m1} = \tau_u f_R f_A\left\{1 + \frac{1.0}{R_{\mathrm{t}}^{3/4}f_R f_A}\sqrt{\frac{2R}{Pr}}\exp\left[-\left(\frac{R_{tm}}{12.0}\right)^{3/4}\right]\right\} \tag{16}$$

$$\tau_{m2} = \tau_{m1}[1 - f_{\mathrm{w}}(30)] \tag{17}$$

where $f_R = \frac{2R}{0.5+R}$ and $f_A = \left(\frac{2}{1+3.5\sqrt{b_{ij}b_{ij}}}\right)\left[1 + \left(\frac{1+3.5\sqrt{b_{ij}b_{ij}}}{2} - 1\right)f_{\mathrm{w}}(26)\right]$ are model functions. The model function $f_{\mathrm{w}}(\xi)$ is the wall-reflection function (Nagano and Hattori, 2002) as follows:

$$f_{\mathrm{w}}(\xi) = \exp\left[-\left(\frac{R_{tm}}{\xi}\right)^2\right] \tag{18}$$

where the modified Reynolds number R_{tm} in the model function $f_{\mathrm{w}}(\xi)$ is introduced as follows:

$$R_{tm} = \frac{C_{tm}n^*R_{\mathrm{t}}^{1/4}}{C_{tm}R_{\mathrm{t}}^{1/4} + n^*} \tag{19}$$

where C_{tm} is model constant set at 1.3×10^2.

3.2. Evaluation of derived NLEDHM

Evaluations of the derivative expression (NLHNB model) in Eq. (14) are conducted using DNS databases under wall-bounded, buoyancy-affected turbulent flow in a heated plane channel with unstable or stable stratification as indicated in Fig. 1(a) (Iida and Kasagi, 1997; $Gr = 1.3 \times 10^6$ where $\Delta\Theta = \Theta_H - \Theta_C$ and $Re_\tau = 150$ (unstable case); Iida et al., 2002; $Gr = 4.4 \times 10^6$ and $Re_\tau = 150$ (stable case)) and in a vertical heated plane channel shown as Fig. 1(b) (Kasagi and Nishimura, 1997; $Gr = 9.6 \times 10^5$ and $Re_\tau = 150$). In the case of a vertical plane channel, the streamwise turbulent heat-flux, $\overline{u\theta}$, appears clearly in both the transport equations of turbulence energy and its dissipation rate. Also, the turbulent heat-flux is included in the modeled expression of Reynolds stress for the buoyancy-affected flow as described later. Therefore, the streamwise turbulent heat-flux should also be predicted exactly by the model. Fig. 2 shows the results of assessments for the NLHNB model of the streamwise turbulent heat-flux in the vertical heated plane channel flow. It can be seen that the NLHNB model underpredicts streamwise turbulent heat-flux. Also, in cases of heated plane channel with unstable or stable stratification, obviously, underpredictions of turbulent heat-flux are observed as shown in Fig. 3. On the other hand, the wall-normal turbulent heat-flux predicted by the NLHNB model is shown in Figs. 4 and 5. It is clear that the wall-normal turbulent heat-flux is not reproduced accurately in these cases. Consequently, we improve the modeled expression of turbulent heat-flux to predict wall-bounded buoyancy-affected turbulent flows.

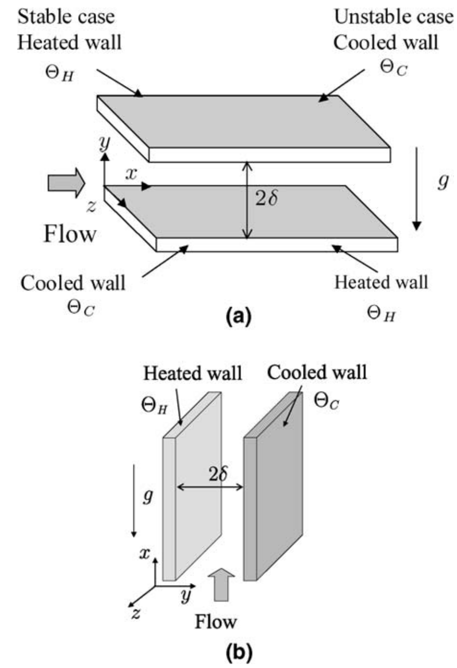

Fig. 1. Flow geometry for model evaluation: (a) heated stable/unstable plane channel, (b) heated vertical plane channel.

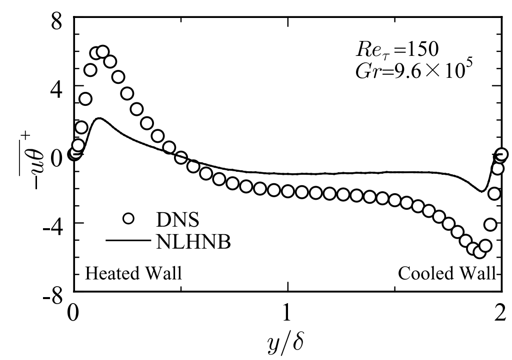

Fig. 2. Evaluation result for modeled streamwise turbulent heat-flux (vertical heated plane channel flow).

3.3. Evaluation of existing NLEDM

In the velocity field, since the NLEDM reflecting buoyancy effect was derived by So et al. (2002) in two-dimen-

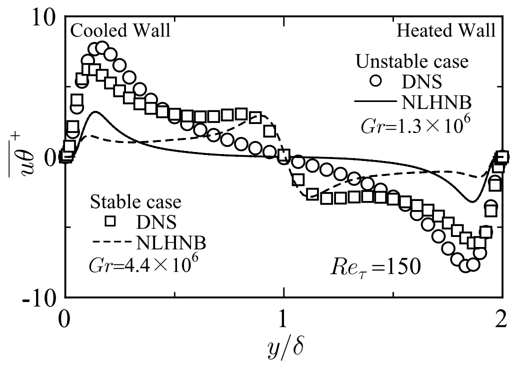

Fig. 3. Evaluation result for modeled streamwise turbulent heat-flux (stable/unstable heated plane channel flows).

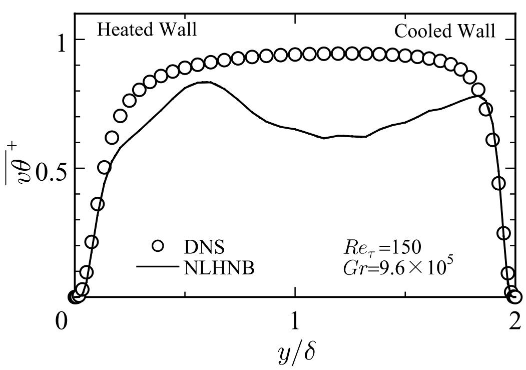

Fig. 4. Evaluation result for modeled wall-normal turbulent heat-flux (vertical heated plane channel flow).

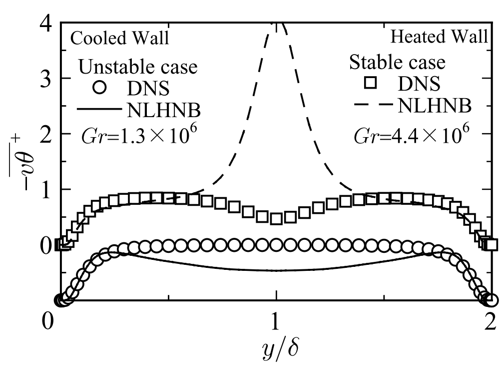

Fig. 5. Evaluation result for modeled wall-normal turbulent heat-flux (stable/unstable heated plane channel flows).

sional buoyant flows (hereinafter referred to as the SVJZG-O model), we evaluate whether or not NLEDM is applicable in wall-bounded buoyancy-affected turbulent flows. The SVJZG-O model is given as follows:

$$\mathbf{b}^* = Q^{(1)}\mathbf{S}^* + Q^{(2)}(\mathbf{S}^*\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{S}^*) + Q^{(3)}\left(\mathbf{S}^{*2} - \frac{1}{3}\{\mathbf{S}^{*2}\}\mathbf{I}\right) + Q^{(4)}\mathbf{f}^* + Q^{(5)}(\mathbf{f}^*\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{f}^*) \tag{20}$$

where $\mathbf{b}^* = b^*_{ij} = \left(\frac{C_3-2}{C_2-\frac{4}{3}}\right)b_{ij}$, $\mathbf{S}^* = S^*_{ij} = \frac{1}{2}g\tau(2-C_3)S_{ij}$, $\boldsymbol{\Omega}^* = \Omega^*_{ij} = \frac{1}{2}g\tau(2-C_4)\Omega_{ij}$ and $\mathbf{f}^* = f^*_{ij} = G_{ij}/G_k - \delta^{(2d)}_{ij}$ are the nondimensional tensors, respectively, $\mathbf{I} = \delta_{ij}$ is the identity tensor, $G_{ij} = \beta(g_j\overline{u_i\theta} + g_i\overline{u_j\theta})$ is the buoyant term in the transport equation of Reynolds stress, and $\delta^{(2d)}_{ij}$ is the two-dimensional tensor first proposed by Pope (1975). Model coefficients $Q^{(1)}$–$Q^{(5)}$ in Eq. (20) are given as follows:

$$Q^{(1)} = \frac{1}{D_1}\left[1 - \frac{1}{3D_2}(\{\mathbf{f}^*\mathbf{S}^*\} + 2\{\boldsymbol{\Omega}^*\mathbf{S}^*\mathbf{f}^*\})G^* - \frac{G^*}{3}\right], \quad Q^{(2)} = Q^{(1)} \tag{21}$$

$$Q^{(3)} = -2\frac{1}{D_1}Q^{(1)} - \frac{1}{D_2\{\mathbf{S}^{*2}\}}(\{\mathbf{f}^*\mathbf{S}^*\} + 2\{\boldsymbol{\Omega}^*\mathbf{S}^*\mathbf{f}^*\})G^* + \frac{1}{\{\mathbf{S}^{*2}\}}G^* \tag{22}$$

$$Q^{(4)} = \frac{1}{2D_2}G^*, \quad Q^{(5)} = \frac{1}{2D_2}G^* \tag{23}$$

$$D_1 = -\left(1 - \frac{2}{3}\{\mathbf{S}^{*2}\} - 2\{\boldsymbol{\Omega}^{*2}\}\right), \quad D_2 = -(1 - 2\{\boldsymbol{\Omega}^{*2}\}) \tag{24}$$

$$G^* = \tau\frac{(C_5-1)(C_3-2)}{2(C_2-\frac{4}{3})k}G_k, \quad g = \left(\frac{1}{2}C_1 + \frac{P_k}{\varepsilon} + \frac{G_k}{\varepsilon} - 1\right)^{-1}, \quad \tau = \frac{k}{\varepsilon} \tag{25}$$

where $C_1 = 3.4$, $C_2 = 0.36$, $C_3 = 1.25$, $C_4 = 0.4$ and $C_5 = 0.3$ are model constants.

The SVJZG-O model is assessed to be identical with cases for thermal field using DNS databases. Fig. 6 shows a result of evaluation in the vertical heated plane channel flow. Note that the dashed line indicates the result of the SVJZG-O model and the solid line shows the modified model (SVJZG-M model). Since Eq. (24) often gives a negative value in parentheses at large $\{\boldsymbol{\Omega}^{*2}\}$, the model overpredicts remarkably near the wall as shown in Fig. 6. Thus, to avoid the overprediction, the functions are modified as $D_1 = -\left(1 - \frac{2}{3}\mathbf{S}^* + 2\{\boldsymbol{\Omega}^{*2}\}\right)$ and $D_2 = -(1 + 2\{\boldsymbol{\Omega}^{*2}\})$. The SVJZG-M model is also evaluated in an unstable heated plane channel flow as shown in Fig. 7. In this case, the model obviously gives an overprediction near the wall. From these evaluations, the predictions are improved in most parts of the channel, but the near-wall behavior of model prediction should be carefully corrected.

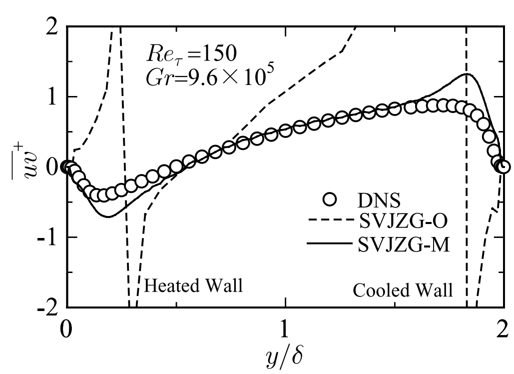

Fig. 6. Evaluation result for modeled Reynolds shear stress (vertical heated plane channel flow).

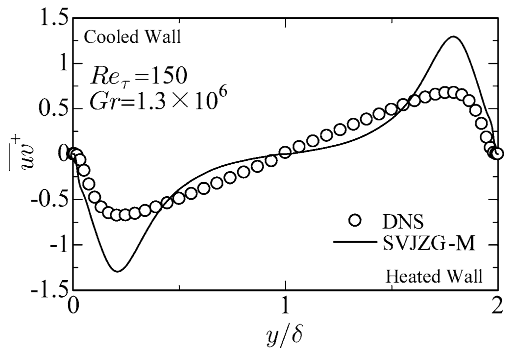

Fig. 7. Evaluation result for modeled Reynolds shear stress (unstable heated plane channel flow).

4. Newly proposed NLEDHM and NLEDM

4.1. Modification of NLEDHM

So far, the modeling for the dissipation term, $\varepsilon_{j\theta}$, in the transportation of turbulent heat-flux has been omitted, because the dissipation does not influence the prediction of wall-normal heat-flux in the wall-bounded flow. However, the dissipation affects predicting the streamwise heat-flux in the wall-bounded flow. Therefore, we introduce the effect of the dissipation-rate into the modeling for Φ_{ij} and $\varepsilon_{j\theta}$ as follows:

$$\Phi_{j\theta}-\varepsilon_{j\theta}=-C_{t1}\frac{\overline{u_j\theta}}{\tau_u}+C_{t2}\overline{u_k\theta}\frac{\partial\overline{U}_j}{\partial x_k}+C_{t3}\overline{u_k\theta}\frac{\partial\overline{U}_k}{\partial x_j}+C_{t5}g_i\beta k_\theta-C_{t4}\overline{u_iu_j}A_{ik}\frac{\partial\overline{\Theta}}{\partial x_k}-C_{t6}A_{jk}g_k\beta k_\theta \quad (26)$$

where $A_{ik}(=\overline{u_iu_k}/k)$ is the nondimensional Reynolds stress tensor. The $\partial\overline{\Theta}/\partial x_k$-related term is included, because the production of $\varepsilon_{j\theta}$ is strongly affected by the temperature gradient. Note that the adopted buoyant term is proposed by Craft et al. (1996). Thus, substituting Eq. (26) into Eq. (11), we can obtain the following equation:

$$\frac{\mathrm{D}a_j^*}{\mathrm{D}t}=\frac{1}{\sqrt{k}\sqrt{k_\theta}}\frac{1}{C_{\theta1}\tau_u}\left[-\overline{u_j\theta}-C_{\theta1}\tau_u\overline{u_ju_k}\frac{\partial\overline{\Theta}}{\partial x_k}-C_{\theta2}\tau_u\overline{u_k\theta}S_{jk}-C_{\theta3}\tau_u\overline{u_k\theta}\Omega_{jk}-C_{\theta4}\tau_u\overline{u_iu_j}A_{ik}\frac{\partial\overline{\Theta}}{\partial x_k}-C_{\theta5}\tau_ug_j\beta k_\theta-C_{\theta6}\tau_uA_{jk}g_k\beta k_\theta\right]-\frac{1}{2}\frac{\overline{u_j\theta}}{\sqrt{k}\sqrt{k_\theta}}\left[\frac{\varepsilon}{k}\left(\frac{P_k}{\varepsilon}+\frac{G_k}{\varepsilon}-1\right)+\frac{\varepsilon_\theta}{k_\theta}\left(\frac{P_{k_\theta}}{\varepsilon_\theta}-1\right)\right] \quad (27)$$

where $C_{\theta1}=1/C_{t1}$, $C_{\theta2}=(1-C_{t2}-C_{t3})/C_{t1}$, $C_{\theta3}=(1-C_{t2}+C_{t3})/C_{t1}$, $C_{\theta4}=C_{t4}/C_{t1}$, $C_{\theta5}=(2-C_{t5})/C_{t1}$ and $C_{\theta6}=C_{t6}/C_{t1}$ are model constants. Also, under the local equilibrium state, $\mathrm{D}a_j^*/\mathrm{D}t=0$, the following relation is derived:

$$\frac{\overline{u_j\theta}}{\sqrt{k}\sqrt{k_\theta}}\frac{1}{\tau_u}\left[1+\frac{C_{\theta1}}{2}\left(\frac{P_k}{\varepsilon}+\frac{G_k}{\varepsilon}-1\right)+\frac{C_{\theta1}}{2R}\left(\frac{P_{k_\theta}}{\varepsilon_\theta}-1\right)\right]=-(\delta_{jk}+3b_{jk})\frac{2}{3}\frac{C_{\theta1}k}{\sqrt{k}\sqrt{k_\theta}}\frac{\partial\overline{\Theta}}{\partial x_k}-(C_{\theta2}S_{jk}+C_{\theta3}\Omega_{jk})\frac{\overline{u_k\theta}}{\sqrt{k}\sqrt{k_\theta}}-(\delta_{ij}+3b_{ij})\frac{2}{3}\frac{kC_{\theta4}}{\sqrt{k}\sqrt{k_\theta}}A_{ik}\frac{\partial\overline{\Theta}}{\partial x_k}-\frac{C_{\theta5}}{\sqrt{k}\sqrt{k_\theta}}g_j\beta k_\theta-\frac{C_{\theta6}}{\sqrt{k}\sqrt{k_\theta}}A_{jk}g_k\beta k_\theta \quad (28)$$

Here, instead of the characteristic time-scale of the left-hand side of Eq. (28), the mixed time-scale τ_m is introduced for considering the wall effect.

$$\frac{\tau_u}{1+\frac{C_{\theta1}}{2}\left(\frac{P_k}{\varepsilon}+\frac{G_k}{\varepsilon}-1\right)+\frac{C_{\theta1}}{2R}\left(\frac{P_{k\theta}}{\varepsilon_\theta}-1\right)}\to\tau_m \quad (29)$$

In this study, the mixed time-scale is modeled as follows (Nagano and Shimada, 1996):

$$\tau_m=\left\{\frac{2R}{2R+C_m}+\sqrt{\frac{2R}{Pr}}\frac{b_{\lambda1}}{R_\mathrm{t}^{3/4}}\exp\left[\left(-\frac{R_{tm}}{b_{\lambda2}}\right)^{3/4}\right]\right\}\tau_u \quad (30)$$

In order to derive the NLEDHM, the following nondimensional quantities are introduced:

$$b_{jk}^*=3b_{jk},\quad \Theta_k^*=\frac{2}{3}\frac{C_{\theta1}k\tau_m}{\sqrt{k}\sqrt{k_\theta}}\frac{\partial\overline{\Theta}}{\partial x_k},\quad S_{jk}^*=C_{\theta2}\tau_mS_{jk},\quad \Omega_{jk}^*=C_{\theta3}\tau_m\Omega_{jk},\quad T_k^*=\frac{2}{3}\frac{C_{\theta4}k\tau_m}{\sqrt{k}\sqrt{k_\theta}}\frac{\partial\overline{\Theta}}{\partial x_k},\quad G_j^*=\frac{2}{3}\frac{C_{\theta5}\tau_m}{\sqrt{k}\sqrt{k_\theta}}g_j\beta k_\theta,\quad F_k^*=\frac{2}{3}\frac{C_{\theta6}\tau_m}{\sqrt{k}\sqrt{k_\theta}}g_k\beta k_\theta \quad (31)$$

Using Eqs. (31), Eq. 30 is transformed into the following simple form:

$$a_j^*=-(\delta_{jk}+b_{jk}^*)\overline{\Theta}_k^*-a_k^*(S_{jk}^*+\Omega_{jk}^*)-(\delta_{ij}+b_{ij}^*)A_{ik}\overline{T}_k^*-G_j^*-A_{jk}F_k^* \quad (32)$$

Finally, the following explicit form regarding a_j^* can be obtained from Eq. (32):

$$a_j^*=\frac{1}{1+\frac{1}{2}(\Omega^{*2}-S^{*2})}\{[-(\delta_{jk}+b_{jk}^*)+(\delta_{\ell k}+b_{\ell k}^*)(S_{j\ell}^*+\Omega_{j\ell}^*)]\Theta_k^*+[-(\delta_{jk}+b_{jk}^*)A_{ik}+(\delta_{\ell i}+b_{\ell i}^*)(S_{j\ell}^*+\Omega_{j\ell}^*)A_{ik}]T_k^*-[\delta_{jk}-(S_{jk}^*+\Omega_{jk})]G_k^*-[\delta_{j\ell}-(S_{j\ell}^*+\Omega_{j\ell})]A_{\ell k}F_k^*\} \quad (33)$$

where $\Omega^{*2}=\Omega_{ij}^*\Omega_{ij}^*$ and $S^{*2}=S_{ij}^*S_{ij}^*$.

Eq. (33) can be rewritten in the dimensional form as follows:

$$\overline{u_j\theta}=-\alpha_{jk}^t\frac{\partial\overline{\Theta}}{\partial x_k}+\frac{\tau_m^2}{f_\mathrm{RT}}(C_{\theta1}\overline{u_\ell u_k}+C_{\theta5}\overline{u_\ell u_i}A_{ik})(C_{\theta2}S_{j\ell}+C_{\theta3}\Omega_{j\ell})\frac{\partial\overline{\Theta}}{\partial x_k}-\frac{2C_{\theta6}\tau_mg_k\beta k_\theta}{f_\mathrm{RT}}[\delta_{jk}-\tau_m(C_{\theta2}S_{jk}+C_{\theta3}\Omega_{jk})]-\frac{2C_{\theta7}\tau_mA_{\ell k}g_k\beta k_\theta}{f_\mathrm{RT}}[\delta_{j\ell}-\tau_m(C_{\theta2}S_{j\ell}+C_{\theta3}\Omega_{j\ell})] \quad (34)$$

where the anisotropic eddy diffusivity for heat is given as

Table 1
Model constants and functions of proposed NLEDHM

C_{tm}	R_{tm}	$f_w(\xi)$	$C_{\theta1}$	$C_{\theta2}$	$C_{\theta3}$	$C_{\theta4}$
1.3×10^2	$\dfrac{C_{tm}n^* R_t^{1/4}}{C_{tm}R_t^{1/4}+n^*}$	Eq. (18)	$0.14[1-f_w(40)]$	0.05	0.11	0.3
$C_{\theta5}$	$C_{\theta6}$	$C_{\theta7}$	C_m	$B_{\lambda1}$	τ_u	τ_θ
$0.3[1-f_w(40)]^2$	$0.2[1-f_w(20)]$	$0.2[1-f_w(20)]$	$0.25/Pr^{1/4}$	$\dfrac{1+2Pr}{20Pr^{0.4}}$	$\dfrac{k}{\varepsilon}$	$\dfrac{k_\theta}{\varepsilon_\theta}$
R	τ_m					
$\dfrac{\tau_\theta}{\tau_u}$	$\left\{\dfrac{2R}{R+C_m}+\sqrt{\dfrac{2R}{Pr}}\dfrac{30}{R_t^{3/4}}\exp\left[-\left(\dfrac{R_{tm}}{B_{\lambda1}}\right)^{3/4}\right]\right\}\tau_u$					

$$\alpha^t_{jk}=\frac{\tau_m}{f_{RT}}(C_{\theta1}\overline{u_j u_k}+C_{\theta4}\overline{u_i u_j}A_{ik}) \tag{35}$$

$$f_{RT}=1+\frac{1}{2}\{\tau_m[1-f_w(40)]\}^2(C_{\theta3}\Omega^2-C_{\theta2}S^2) \tag{36}$$

where model constants and functions in Eq. (34) are indicated in Table 1.

To model the transport equation for thermal field, the turbulent diffusion terms in Eqs. (8) and (9) are modeled by GGDH modeling (Nagano and Hattori, 2003):

$$T_{k\theta}=\frac{\partial}{\partial x_j}\left(C_h f_{\theta1}\frac{k}{\varepsilon}f_R\overline{u_j u_\ell}\frac{\partial k_\theta}{\partial x_\ell}\right) \tag{37}$$

$$T_{\varepsilon_\theta}=\frac{\partial}{\partial x_j}\left(C_\phi f_{\theta2}\frac{k}{\varepsilon}f_R\overline{u_j u_\ell}\frac{\partial \varepsilon_\theta}{\partial x_\ell}\right) \tag{38}$$

Also, model constants and functions in Eqs. (8) and (9) are indicated in Table 2.

4.2. Modification of NLEDM

The transport equation of Reynolds stress with the buoyant term is given as follows:

$$\frac{D\overline{u_i u_j}}{Dt}=D_{ij}+T_{ij}+P_{ij}+G_{ij}+\Phi_{ij}-\varepsilon_{ij} \tag{39}$$

where D_{ij} is a molecular diffusion term, T_{ij} is a turbulent and pressure diffusion term, $P_{ij}=-\overline{u_i u_k}(\partial\overline{U}_j/\partial x_k)-\overline{u_j u_k}(\partial\overline{U}_i/\partial x_k)$ is a production term, $G_{ij}=-\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta})$ is a buoyant term, Φ_{ij} is a pressure–strain correlation term and ε_{ij} is a dissipation term, respectively.

Introducing the Reynolds stress anisotropy tensor $b_{ij}=\overline{u_i u_j}/2k-\delta_{ij}/3$ and neglecting the diffusive effect, the following relation is derived with Eqs. (6) and (39):

$$\frac{Db_{ij}}{Dt}=\frac{1}{2k}(P_{ij}+G_{ij}+\Phi_{ij}-\varepsilon_{ij})-\frac{b_{ij}+\delta_{ij}/3}{k}(P_k+G_k-\varepsilon) \tag{40}$$

In the local equilibrium state, since the relation $Db_{ij}/Dt=0$ holds, Eq. (40) yields the following relation:

$$(P_{ij}+G_{ij}+\Phi_{ij}-\varepsilon_{ij})=2\left(b_{ij}+\frac{\delta_{ij}}{3}\right)(P_k+G_k-\varepsilon) \tag{41}$$

Using the form $\varepsilon_{ij}=\frac{2}{3}\varepsilon\delta_{ij}+{}_D\varepsilon_{ij}$ of the dissipation term (Gatski and Speziale, 1993) for Eq. (41), we can obtain

$$\begin{aligned}(P_k+G_k-\varepsilon)b_{ij}=&-\frac{2}{3}kS_{ij}-k\left(b_{ik}S_{jk}+b_{jk}S_{ik}-\frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right)\\&-k(b_{ik}\Omega_{jk}+b_{jk}\Omega_{ik})+\frac{1}{2}\Pi_{ij}\\&+\frac{1}{2}\left(G_{ij}-\frac{2}{3}\delta_{ij}G_k\right)\end{aligned} \tag{42}$$

where $\Pi_{ij}=\Phi_{ij}-{}_D\varepsilon_{ij}$, and the modeled Π_{ij} is employed as the following general linear model with the buoyant term.

$$\begin{aligned}\Pi_{ij}=&-C_1\varepsilon b_{ij}+C_2kS_{ij}\\&+C_3k\left(b_{ik}S_{jk}+b_{jk}S_{ik}-\frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right)\\&+C_4k(b_{ik}\Omega_{jk}+b_{jk}\Omega_{ik})-C_5\left(G_{ij}-\frac{2}{3}\delta_{ij}G_k\right)\end{aligned} \tag{43}$$

where C_1–C_5 are model constants.

Table 2
Model constants and functions of transport equations for k_θ and ε_θ

C_{P1}	C_{P2}	C_{P3}	$C_{\varepsilon2}$	f_{P1}	f_{P2}	f_{P3}	C_{D1}	f_{D1}
$0.85\,(R+0.4/Pr^{1/4})$	0.64	1.2	1.9	$1-f_w(1)$	1.0	1.0	$1-f_w(4)$	1.0
$C_{D2}f_{D2}$			$C^*_{D2}f^*_{D2}$					
$C^*_{D2}f^*_{D2}\left[1+C^*_{D3}f^*_{D3}\sqrt{R_t}\left(1+C_\tau\sqrt{\dfrac{Pr}{R}}\right)\right]$			$(C_{\varepsilon2}f_{\varepsilon2}-1)\left\{1-\exp\left[-\left(C_{\varepsilon t}\dfrac{R_{tm}}{5}\right)^2\right]\right\}$					
$f_{\varepsilon2}$	C^*_{D3}		f^*_{D3}	C_τ	$C_{\varepsilon t}$	$f_{\theta1}$	$f_{\theta2}$	
$1-0.3\exp\left[-\left(\dfrac{R_t}{6.5}\right)^2\right]$	0.025		$f_w(30)$	3.0	$1+Pr^{1.5}$	$1+5f_w(5)$	$1+20f_w(10)$	
C_h	C_ϕ		f_R					
0.20	0.25		$2R/(R+0.5)$					

Substituting Eq. (43) into (41) and introducing nondimensional quantities, we can obtain the following relation:

$$b^*_{ij} = -S^*_{ij} - \left(b^*_{ik}S^*_{jk} + b^*_{jk}S^*_{ik} - \frac{2}{3}b^*_{k\ell}S^*_{k\ell}\delta_{ij}\right) + b^*_{ik}\Omega^*_{kj} + b^*_{jk}\Omega^*_{ki} - f_{ij} \quad (44)$$

where

$$\left.\begin{aligned} &S^*_{ij} = \tfrac{1}{2}g\tau(2-C_3)S_{ij}, \quad \Omega^*_{ij} = \tfrac{1}{2}g\tau(2-C_4)\Omega_{ij}, \\ &b^*_{ij} = \left(\tfrac{C_3-2}{C_2-\frac{4}{3}}\right)b_{ij}, \quad \tau = \tfrac{k}{\varepsilon} \\ &f^*_{ij} = g\tau\tfrac{(C_5-1)(C_3-2)}{2\left(C_2-\frac{4}{3}\right)k}\left(G_{ij} - \tfrac{2}{3}G_k\delta_{ij}\right), \\ &g = \left(\tfrac{1}{2}C_1 + \tfrac{P_k}{\varepsilon} + \tfrac{G_k}{\varepsilon} - 1\right)^{-1} \end{aligned}\right\} \quad (45)$$

Eq. (45) can be written in the matrix form as

$$\mathbf{b}^* = -\mathbf{S}^* - \left(\mathbf{b}^*\mathbf{S}^* + \mathbf{S}^*\mathbf{b}^* - \frac{2}{3}\{\mathbf{b}^*\mathbf{S}^*\}\mathbf{I}\right) + \mathbf{b}^*\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{b}^* - \mathbf{f}^* \quad (46)$$

In order to derive an explicit form of $\mathbf{b}^*$ from Eq. (46), the integrity basis, $\mathbf{b}^* = \sum_\lambda Q^{(\lambda)}\mathbf{T}^{(\lambda)}$ first proposed by Pope (1975), is used with the following seven basis tensors (So et al., 2002):

$$\left.\begin{aligned} &\mathbf{T}^{(1)} = \mathbf{S}^*, && \mathbf{T}^{(5)} = \mathbf{f}^*\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{f}^* \\ &\mathbf{T}^{(2)} = \mathbf{S}^*\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{S}^*, && \mathbf{T}^{(6)} = \mathbf{f}^{*2} - \tfrac{1}{3}\{\mathbf{f}^{*2}\}\mathbf{I} \\ &\mathbf{T}^{(3)} = \mathbf{S}^{*2} - \tfrac{1}{3}\{\mathbf{S}^{*2}\}\mathbf{I}, && \mathbf{T}^{(7)} = \mathbf{f}^*\mathbf{S}^* + \mathbf{S}^*\mathbf{f}^* - \tfrac{2}{3}\{\mathbf{f}^*\mathbf{S}^*\}\mathbf{I} \\ &\mathbf{T}^{(4)} = \mathbf{f}^* \end{aligned}\right\} \quad (47)$$

Substituting Eq. (47) into $\mathbf{b}^* = \sum_\lambda Q^{(\lambda)}\mathbf{T}^{(\lambda)}$ gives

$$\mathbf{b}^* = Q^{(1)}\mathbf{S}^* + Q^{(2)}(\mathbf{S}^*\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{S}^*) + Q^{(3)}\left(\mathbf{S}^{*2} - \frac{1}{3}\{\mathbf{S}^{*2}\}\mathbf{I}\right) + Q^{(4)}\mathbf{f}^* + Q^{(5)}(\mathbf{f}^*\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{f}^*) + Q^{(6)}\left(\mathbf{f}^{*2} - \frac{1}{3}\{\mathbf{f}^{*2}\}\mathbf{I}\right) + Q^{(7)}\left(\mathbf{f}^*\mathbf{S}^* + \mathbf{S}^*\mathbf{f}^* - \frac{2}{3}\{\mathbf{f}^*\mathbf{S}^*\}\mathbf{I}\right) \quad (48)$$

On the other hand, substituting the scalar functions H and J related with $\mathbf{T}^\lambda$ indicated by Pope (1975), $\mathbf{T}^{(\lambda)}\mathbf{S}^* + \mathbf{S}^*\mathbf{T}^{(\lambda)} - \frac{2}{3}\{\mathbf{T}^{(\lambda)}\mathbf{S}^*\}\mathbf{I} = \sum_\gamma H_{\lambda\gamma}\mathbf{T}^{(\gamma)}$ and $\mathbf{T}^{(\lambda)}\mathbf{\Omega}^* - \mathbf{\Omega}^*T^{(\lambda)} = \sum_\gamma J_{\lambda\gamma}\mathbf{T}^{(\gamma)}$ and the integrity basis into Eq. (46) give the following equation for $\mathbf{T}^{(\lambda)}$:

$$\sum_\lambda Q^{(\lambda)}\mathbf{T}^{(\lambda)} = -\sum_\lambda \delta_{1\lambda}\mathbf{T}^{(1)} - \left[\sum_\lambda Q^{(\lambda)}\left(\sum_\lambda H_{\lambda\gamma}\mathbf{T}^{(\lambda)}\right)\right] + \sum_\lambda Q^{(\lambda)}\left(\sum_\lambda J_{\lambda\gamma}\mathbf{T}^{(\lambda)}\right) - \sum_\lambda \delta_{4\lambda}\mathbf{T}^{(4)} \quad (49)$$

Eq. (49) can be written as $Q^{(\lambda)} = A^{-1}_{\gamma\lambda}B_\lambda$, where $A_{\gamma\lambda} = -\delta_{\lambda\gamma} - H_{\lambda\gamma} + J_{\lambda\gamma}$, and $B_\lambda = \delta_{1\lambda} + \delta_{4\lambda}$. By using Cayley–Hamilton identities, the matrices $H_{\lambda\gamma}$ and $J_{\lambda\gamma}$ can be determined. Thus, we can obtain $Q^{(\lambda)}$ using Mathematica as follows:

$$\left.\begin{aligned} &Q^{(1)} = \tfrac{1}{D_1}, \ Q^{(2)} = \tfrac{1}{D_1}, \ Q^{(3)} = -\tfrac{2}{D_1}, \ Q^{(4)} = \tfrac{1}{D_2} \\ &Q^{(5)} = \tfrac{1}{D_2}, \ Q^{(6)} = 0, \ Q^{(7)} = \tfrac{1}{D_2} \end{aligned}\right\} \quad (50)$$

where

$$D_1 = -\left(1 - \frac{2}{3}\eta_1 - 2\eta_2\right), \quad D_2 = -(1 - 2\eta_2) \quad (51)$$

with $\eta_1 = (S^*_{ij}S^*_{ij})^{1/2}$ and $\eta_2 = (\Omega^*_{ij}\Omega^*_{ij})^{1/2}$.

Therefore, we can obtain the expression of Reynolds shear stress with buoyant effect for NLEDM as follows:

$$b^*_{ij} = -\frac{3}{3 - 2\eta^2 + 6\zeta^2}\left[S^*_{ij} + (S^*_{ik}\Omega^*_{kj} - \Omega^*_{ik}S^*_{kj}) - 2\left(S^*_{ik}S^*_{kj} - \frac{1}{3}S^*_{mn}S^*_{nm}\delta_{ij}\right)\right] - \frac{1}{1 + 2\zeta^2}\left[f^*_{ij} + (f^*_{ik}\Omega^*_{kj} - \Omega^*_{ik}f^*_{kj}) + \left(f^*_{ik}S^*_{kj} + S^*_{ik}f^*_{kj} - \frac{2}{3}f^*_{mn}S^*_{mn}\delta_{ij}\right)\right] \quad (52)$$

where $\eta = \eta_1 = (S^*_{ij}S^*_{ij})^{1/2}$, $\zeta = \eta_2 = (\Omega^*_{ij}\Omega^*_{ij})^{1/2}$ and $f_{ij} = G_{ij} - (2/3)G_k\delta_{ij}$. Note that in order to avoid an inappropriate value of D_1 and D_2 referring to the assessment results, D_1 and D_2 in Eq. (51) are modeled as $D_1 = (3 - 2\eta^2 + 6\zeta^2)/3$ and $D_2 = 1 + 2\zeta^2$, respectively. Here, the nondimensional forms of b^*_{ij}, S^*_{ij}, Ω^*_{ij} and f^*_{ij} are adopted as follows (Abe et al., 1997; Nagano et al., 1997):

$$b^*_{ij} = C_D b_{ij}, \ S^*_{ij} = C_D\tau S_{ij}, \\ \Omega^*_{ij} = 72C_D\tau\Omega_{ij}, \ f^*_{ij} = C_g(\tau_{mg}/k)f_{ij} \quad (53)$$

Consequently, the newly proposed expressions are as follows:

$$\overline{u_iu_j} = \frac{2}{3}k\delta_{ij} - \frac{2\nu_t}{f_{R1}}S_{ij} - \frac{4C_Dkf_\tau}{f_{R1}}(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}) + \frac{4C_Dkf_\tau}{f_{R1}}\left(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij}\right) - \frac{2C_g\tau_{mg}}{C_Df_{R2}}f_{ij} - \frac{4C_g\tau^2_{mg}}{f_{R2}}(f_{ik}\Omega_{kj} - \Omega_{ik}f_{kj}) + \frac{2C_g\tau^2_{mg}}{f_{R2}}\left(f_{ik}S_{kj} + S_{ik}f_{kj} - \frac{2}{3}f_{mn}S_{mn}\delta_{ij}\right) \quad (54)$$

$$f_{R1} = 1 + \frac{22}{3}(C_D\tau_{R_0})^2\Omega^2 + \frac{2}{3}(C_D\tau_{R_0})^2(\Omega^2 - S^2)f_B \quad (55)$$

$$f_{R2} = 1 + 8(C_D\tau_{R_0})^2\Omega^2 \quad (56)$$

where τ_{mg} is the mixed scale of velocity and thermal fields, and f_τ is the function of characteristic time-scale reflecting wall-limiting behavior (Nagano and Hattori, 2003) as follows:

$$f_\tau = \tau^2_{R_0} + \tau^2_{R_w} \quad (57)$$

where the characteristic time-scale τ_{R_0} is given by ν_t/k, and τ_{R_w} is the wall-reflection time-scale. The other model constants and functions in Eq. (54) are indicated in Table 3.

Table 3
Model constants and functions of proposed NLEDM

C_μ	C_D	C_η	C_{v1}	C_{v2}	C_g	C_λ	τ_{mg}
0.12	0.8	5.0	0.4	1.0×10^2	−0.7	0.1	$C_\lambda f_\lambda \frac{k}{\varepsilon}$

f_μ		$f_{\mathrm{w}}(\xi)$
$[1-f_{\mathrm{w}}(32)]\left\{1+\frac{40}{R_{\mathrm{t}}^{3/4}}\exp\left[-\left(\frac{R_{tm}}{35}\right)^{3/4}\right]\right\}$		Eq. (18)
f_λ		
$[1-f_{\mathrm{w}}(25)]\left\{1+\sqrt{\frac{2R}{Pr}}\frac{15}{R_{\mathrm{t}}^{3/4}}\exp\left[-\left(\frac{R_{tm}}{30}\right)^{3/4}\right]\right\}$		
τ_{R_0}	$\tau_{R_{\mathrm{w}}}$	f_B
$\frac{\nu_t}{k}$	$\sqrt{\frac{1}{6}\frac{f_{R1}/C_D}{f_{S\Omega}}\left(1-\frac{3C_{v1}f_{v2}}{8}\right)}f_{v1}^2$	$1+C_\eta(C_D\tau_{R_0})^2(\Omega^2-S^2)$
$f_{S\Omega}$		S_f
$\frac{\Omega^2}{2}+\frac{S^2}{3}-\left[\left(\sqrt{\frac{S^2}{2}}-\sqrt{\frac{\Omega^2}{2}}\right)f_{\mathrm{w}}(1)\right]^2$		$1+\sqrt{S^2}\tau_u$
f_{v1}		f_{v2}
$\exp\left[-\frac{(R_{tm}/52)^2 S_f}{(R_{tm}/52)^2+S_f}\right]$		$1-\exp\left(-\frac{\sqrt{R_{\mathrm{t}}}}{C_{v2}}\right)$

Table 4
Model constants and functions of transport equations for k and ε

$C_{\varepsilon1}$	$C_{\varepsilon2}$	$C_{\varepsilon3}$	C_s	C_ε
1.45	1.9	1.2	1.4	1.8

f_ε	f_{t1}	f_{t2}
$\left\{1-0.3\exp\left[-\left(\frac{R_{\mathrm{t}}}{6.5}\right)^2\right]\right\}[1-f_{\mathrm{w}}(3.7)]$	$\frac{1+15f_{\mathrm{w}}(10)}{[1-f_{\mathrm{w}}(32)]^{1/2}}$	$\frac{1+10f_{\mathrm{w}}(10)}{[1-f_{\mathrm{w}}(32)]^{1/2}}$

To model the transport equation for the velocity field, the turbulent diffusion terms in Eqs. (6) and (7) are modeled by GGDH modeling (Nagano and Hattori, 2003):

$$T_k = \frac{\partial}{\partial x_j}\left(C_s f_{t1}\frac{\nu_t}{k}\overline{u_j u_\ell}\frac{\partial k}{\partial x_\ell}\right) \tag{58}$$

$$T_\varepsilon = \frac{\partial}{\partial x_j}\left(C_\varepsilon f_{t2}\frac{\nu_t}{k}\overline{u_j u_\ell}\frac{\partial \varepsilon}{\partial x_\ell}\right) \tag{59}$$

Also, the model constants and functions in Eqs. (6) and (7) are indicated in Table 4.

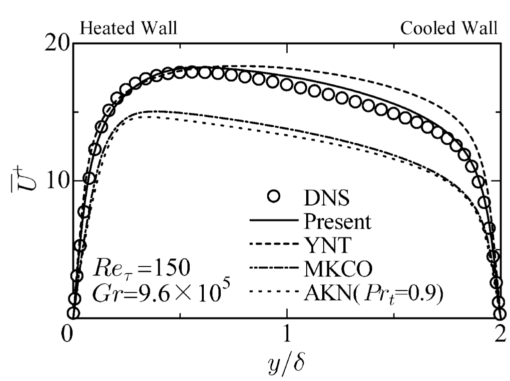

Fig. 8. Distributions of mean velocity in vertical heated plane channel flow.

5. Results and discussion

The evaluations for the newly improved models are shown in Figs. 8–15. The predictions of the conventional

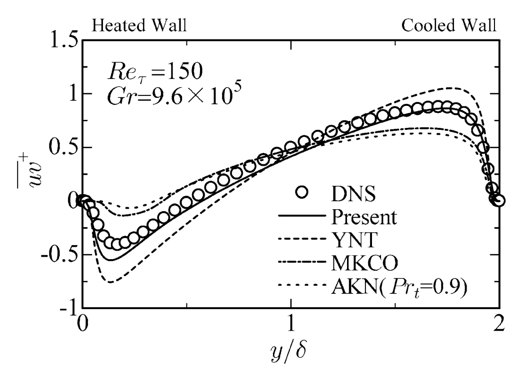

Fig. 9. Distributions of Reynolds shear stress in vertical heated plane channel flow.

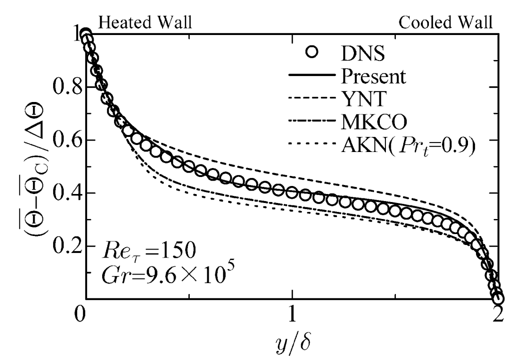

Fig. 10. Distributions of mean temperature in vertical heated plane channel flow.

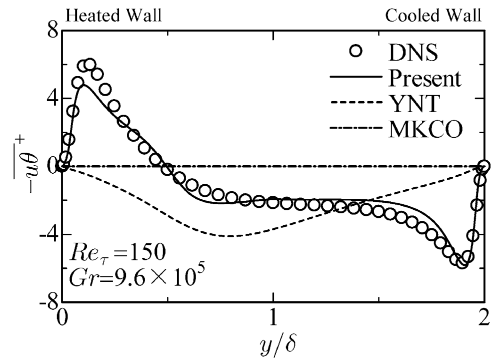

Fig. 11. Distributions of streamwise heat-flux in vertical heated plane channel flow.

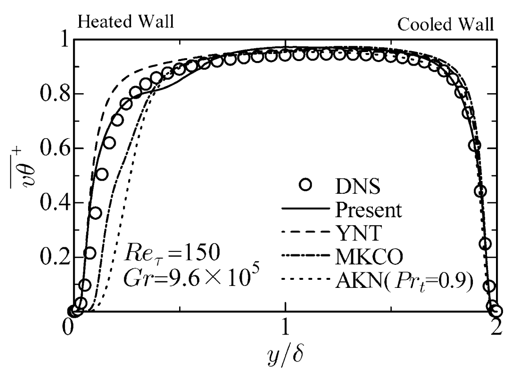

Fig. 12. Distributions of wall-normal heat-flux in vertical heated plane channel flow.

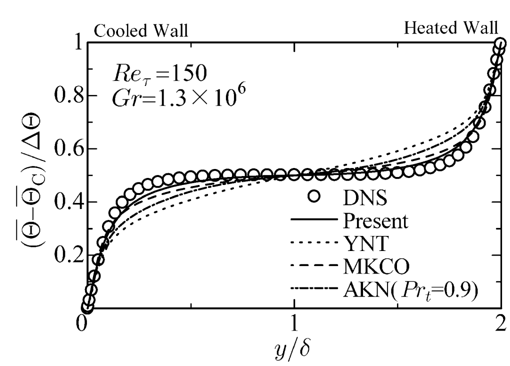

Fig. 13. Distributions of mean temperature in unstable heated plane channel flow.

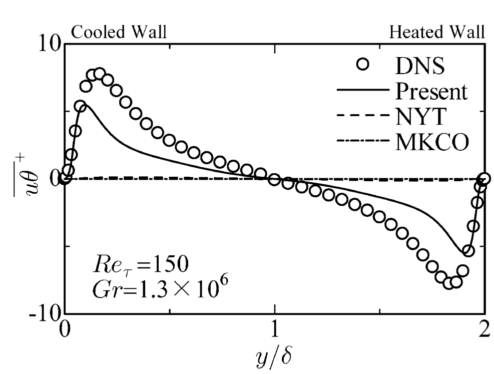

Fig. 14. Distributions of streamwise heat-flux in unstable heated plane channel flow.

eddy viscosity model reflecting the buoyant effect (Yin et al., 1991; YNT model, Murakami et al., 1996; MKCO model) and a constant turbulent Prandtl number ($Pr_t = 0.9$) model are also included in these figures for comparison. The numerical technique used is a finite-volume method (Hattori and Nagano, 1995; Nagano and Hattori, 2003).

In the case of a vertical heated plane channel flow shown as Figs. 8–12, it can be seen that almost all models accu-

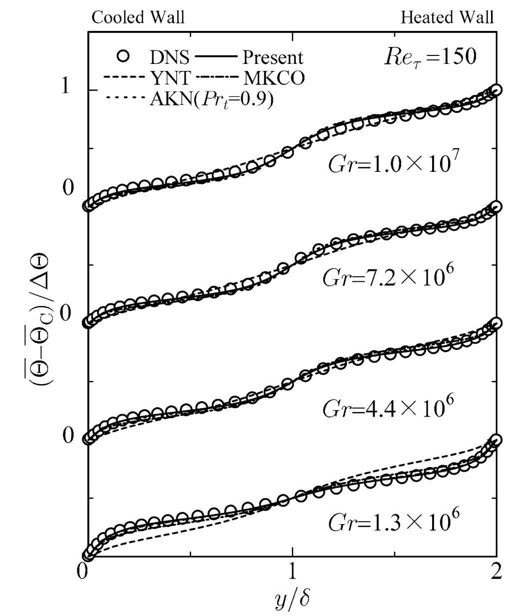

Fig. 15. Distributions of mean temperature in stable heated plane channel flow.

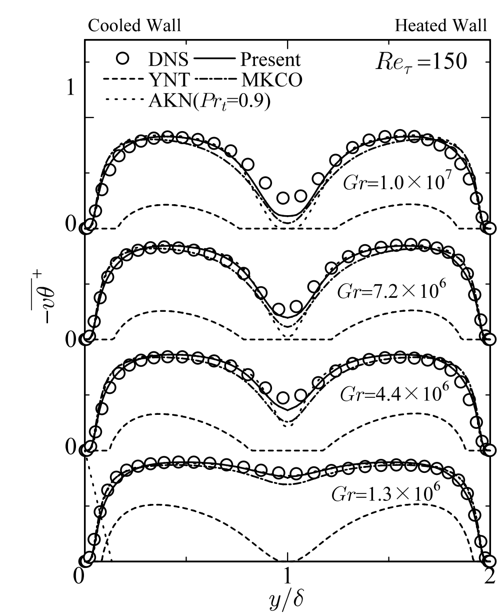

Fig. 16. Distributions of wall-normal turbulent heat-flux in stable heated plane channel flow.

rately predict Reynolds shear stress, mean temperature and turbulent heat-fluxes. In this case, since the streamwise turbulent heat-flux, $\overline{u\theta}$, appears explicitly in the transport equations with the buoyant term, G_k, the adequate prediction of $\overline{u\theta}$ should be required. Only the present model gives an adequate prediction of the streamwise turbulent heat-flux as shown in Fig. 11. Thus, the present models can accurately predict this flow.

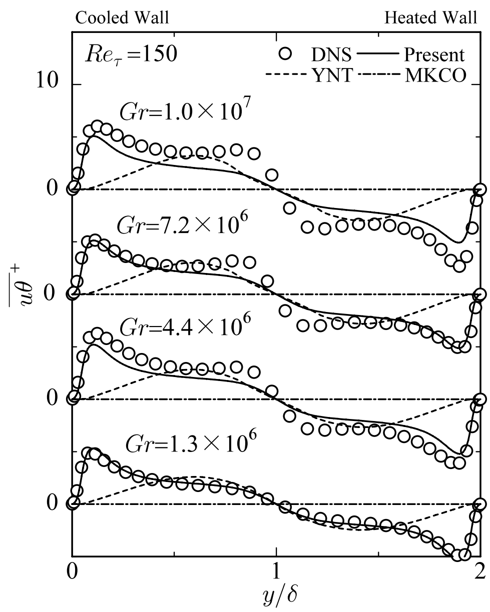

Fig. 17. Distributions of streamwise turbulent heat-flux in stable heated plane channel flow.

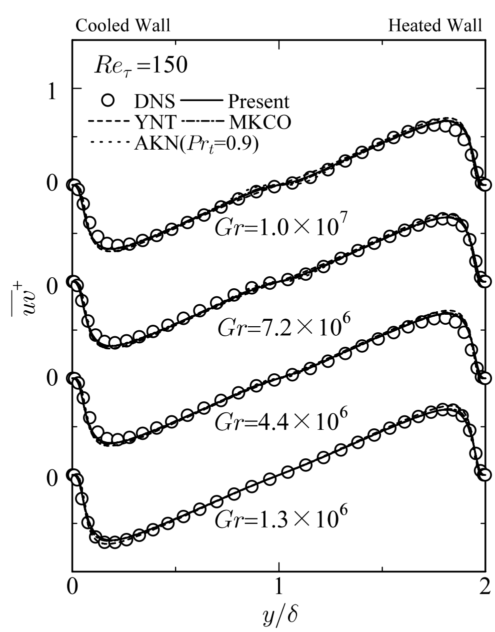

Fig. 18. Distributions of Reynolds shear stress in stable heated plane channel flow.

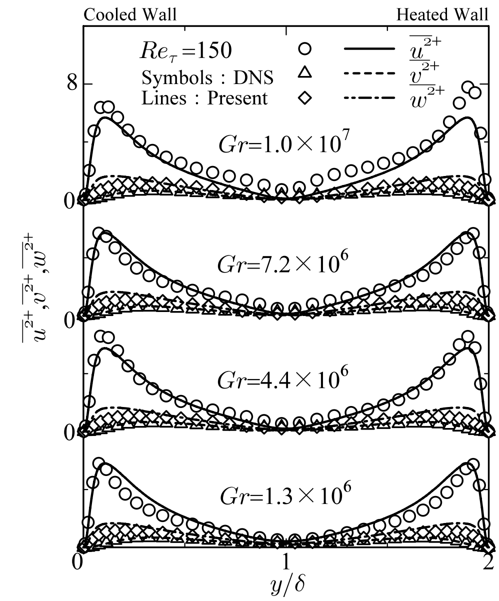

Fig. 19. Distributions of Reynolds normal stress in stable heated plane channel flow.

The proposed model can accurately predict the cases of unstable heated plane channel flows as indicated in Figs. 13 and 14. In this case, since the turbulence energy is also

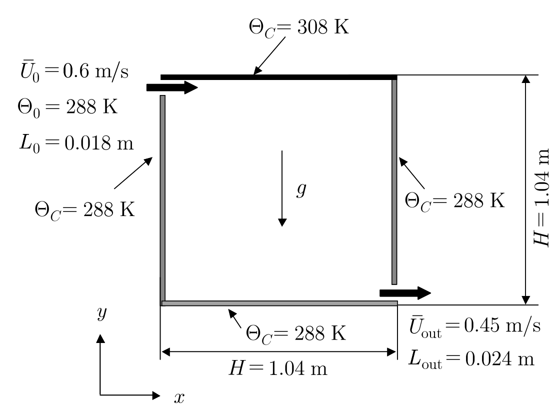

Fig. 20. Two-dimensional enclosed space with supply and exhaust.

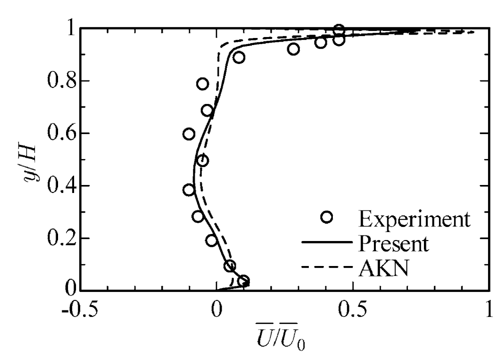

Fig. 21. Predicted profiles of mean velocity, $\overline{U}$ ($x/H = 0.5$).

underpredicted due to the overprediction of ε near the wall, the streamwise Reynolds normal stress, $\overline{u^2}$, gives an underprediction compared with the DNS result (figure not shown here). Thus, the streamwise turbulent heat-flux is slightly underpredicted by the proposed model, but the others cannot predict as shown in Fig. 14.

Next, the proposed model is evaluated in the case of stable heated plane channel flows as shown in Figs. 15–19. In this case, even the linear turbulence models reflecting the

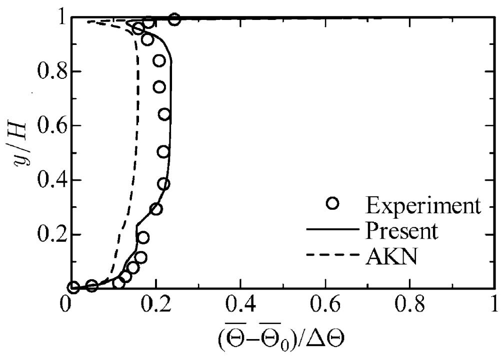

Fig. 22. Predicted profiles of mean temperature at the horizontal center cross-section ($x/H = 0.5$).

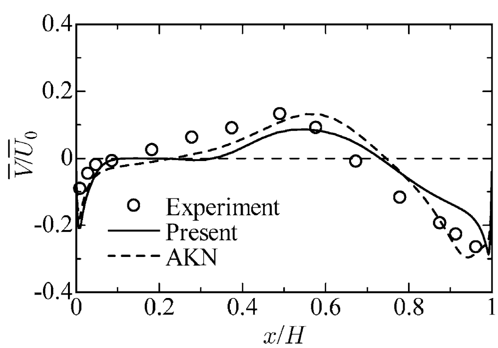

Fig. 23. Predicted profiles of mean velocity, $\overline{V}$, ($y/H = 0.5$).

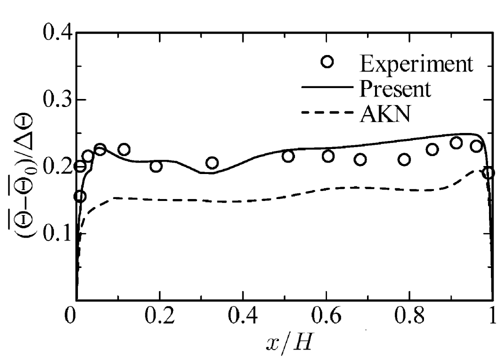

Fig. 24. Predicted profiles of mean velocity at the vertical center cross-section ($y/H = 0.5$).

buoyancy effect except for the $Pr_t = $ constant model give reasonable predictions except for the streamwise turbulent heat-flux. However, the predicted turbulent quantities of the proposed model give better predictions of other models' agreement with DNS results in various Grashhof number flows.

Thus, it is concluded that the present model is available for accurate prediction of wall-shear flows with buoyancy. Note that Reynolds shear stress and the anisotropy of turbulent intensities near the wall are appropriately obtained by the proposed model as indicated in Figs. 18 and 19.

Finally, in order to confirm the model performance, a cavity flow with stable stratification (Blay et al., 1992) is predicted using the proposed model. The flow conditions are shown in Fig. 20, and Archimedes number $Ar = 0.034$ and Reynolds number $Re = 722$ are set. The results of predictions are shown in Figs. 21–24. The AKN model applying buoyant flow calculation which is the linear two-equation heat transfer model is included for comparison. Obviously, the predictions of proposed model indicate good agreement with experimental data.

6. Conclusions

DNS-based evaluations of the modeled expressions for Reynolds stress and turbulent heat-flux for NLEDM and NLEDHM are conducted in wall-shear flows with buoyancy. In particular, it is found that the streamwise turbulent heat-flux, $\overline{u\theta}$, is underpredicted in all cases. In the case of a vertical plane channel, the streamwise turbulent heat-flux, $\overline{u\theta}$, appears clearly in the transport equations for turbulence energy, its dissipation rate and dissipation rate of temperature variance. Also, the turbulent heat-flux is included in the modeled expression of Reynolds stress for the buoyancy-affected flow. Therefore, the streamwise turbulent heat-flux should also be exactly predicted by the model. Thus, we have proposed here a new nonlinear two-equation turbulence models which can satisfactorily predict wall-shear flows with buoyancy.

Acknowledgement

This research was partially supported by Grant-in-Aid for Scientific Research (S), 17106003 from the Ministry of Education, Culture, Sports and Technology (MEXT).

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